

2nd Iranian Combinatorics Olympiad

July 29-30, 2021



Contest problems with solutions

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This booklet is prepared by Sina Ghaseminejad.

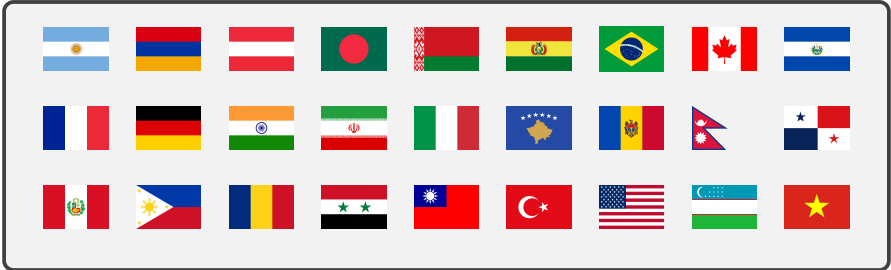
With special thanks to Morteza Saghafian, Afrouz Jabal Ameli, Arian Vahabpour,
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Participating Countries

List of participated nations at the 2nd Iranian Combinatorics Olympiad

Argentina	Armenia	Austria
Bangladesh	Belarus	Bolivia
Brazil	Canada	El Salvador
France	Germany	India
Iran	Italy	Kosovo
Moldova	Nepal	Panama
Peru	Philippines	Romania
Syria	Taiwan	Turkey
United States	Uzbekistan	Vietnam



The 2nd Iranian Combinatorics Olympiad was held on July 29th and 30th , 2021 with over 1600 participants in almost 800 teams from 27 countries. The Problem Selection Committee for 2nd Iranian Combinatorics Olympiad was consisted of



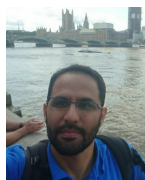
Alireza
Alipour



Abolfazl
Asadi



Amin
Behjati



Seyed Hesam
Firouzi



Yaser
Fouladi



Amir Hossein
Gorzi



Seyed Reza
Hosseini



Afrouz
Jabal Ameli



Morteza
Saghafian

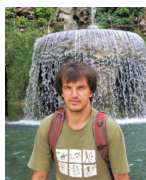


Abbas
Servati

And the International Problem Selection Committee for 2nd Iranian Combinatorics Olympiad was consisted of



Ilya
Bogdanov



Alexader
Polyanskii



Josef
Tkadlec

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First Round

Problems

Problem 1.

What is the maximum number of points we can place on a plane in such a way that for every subset of these points, we would be able to draw a rectangle whose sides are parallel to the axes and every point in the subset is placed inside the rectangle and others are placed outside of it.

($\rightarrow$ p.9)

Problem 2.

There are some players in a 70-day chess tournament. On each day exactly one game will be played between two players (games can also be repeated). The winner of each game, will not play for at most 8 days, and the loser of each game will not play any other games for at least 9 days. Find the minimum possible number of the participants of this tournament.

($\rightarrow$ p.9)

Problem 3.

There are 35 people in a group which are liars or truth-tellers. Liars always tell lies and truth-tellers always tell the truth. For every i , the i -th person has told us that the number of truth-tellers in this group is a divisor of $2i$. Find the number of possible values for the number of truth-tellers in this group.

($\rightarrow$ p.9)

Problem 4.

We have filled k cells of an 8×8 grid with distinct numbers in such a way that the number in each square is greater than at most one filled square with a common corner. Find the maximum possible value for k .

($\rightarrow$ p.9)

Problem 5.

Find the number of ways that we can tile the figure below with dominoes. A tiling is a placement of dominoes that covers all the squares of the board perfectly (i.e. no overlaps, no diagonal placements, no protrusions off the board).

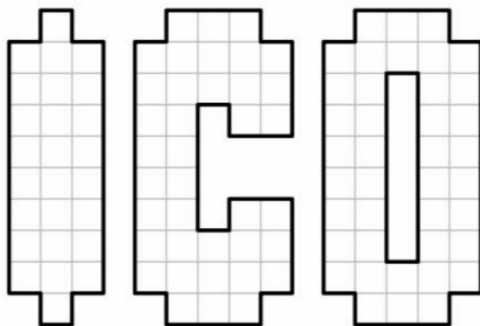


Figure 1: The figure that should be tiled with dominoes.

(→ p.9)

Problem 6.

We have filled a 10×10 grid with numbers $+1$ and -1 . Find the maximum amount of k such that there are exactly k rows with positive sum and exactly k columns with negative sum.

(→ p.9)

Problem 7.

Some mathematicians have participated in a combinatorics competition. Each mathematician can speak several languages. These mathematicians can speak k languages in total. We know that every two mathematicians can talk to each other directly or by a common language mediator among the mathematicians. We also know that if any mathematician leaves this competition, then this fact won't be true anymore. Find the minimum possible amount for k .

(→ p.9)

Problem 8.

There are 1400 boys around a circle forming a regular 1400-gon. We know that k of them are honest (always tell the truth) but we don't know which boys are honest, and the others are dishonest (sometimes tell the truth and sometimes lie). One of them, Mr.X, has a diamond in his pocket whom we want to find. We ask each one of the boys about the circular distance between him and Mr.X. Find the smallest value for k such that we can certainly find Mr.X having all their answers.

($\rightarrow$ p.9)

Problem 9.

Find the number of ways we can color the edges of a labeled complete graph with 5 vertices (vertices are labeled with 1 to 5) with 3 colors in such a way that every vertex has at least one edge with each of the 3 colors.

($\rightarrow$ p.9)

Problem 10.

A 12×13 chessboard (12 rows and 13 columns) is tiled with 1×3 mosaics in such a way that there are exactly k vertical mosaics in every column. Find the number of different possible answers for k . A tiling is a placement of mosaics that covers all the squares of the board perfectly (i.e. no overlaps, no diagonal placements, no protrusions off the board).

($\rightarrow$ p.9)

Problem 11.

We have 100 vertices labeled from 1 to 100 around a circle as in the figure. As you can see, beside the small arcs, there are some extra edges which connect vertex 49 to some other vertices. Find the number of different paths from point 1 to point 100 in this graph.

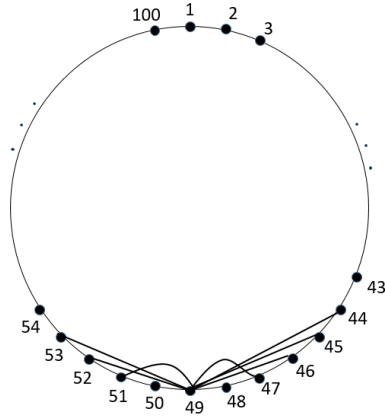


Figure 2: A cycle of length 100, the vertex labeled 49 is connected to vertices 44-48 and 50-53.

(→ p.9)

Problem 12.

There are 30 (not necessarily distinct) real numbers written on a board. We know that no matter how we divide these numbers into 10 groups of size 3, there would be at least 2 groups with equal sums. Find the maximum number of distinct numbers on the board.

(→ p.9)

Problem 13.

Find the maximum number of 3-digit numbers with digits 1 to 4 in such a way that for every two of them like $\overline{a_1a_2a_3}$ and $\overline{b_1b_2b_3}$, there exists an index i such that $a_i + b_i = 5$.

(→ p.9)

Problem 14.

We call a lattice point (x, y) *good* if x, y are natural numbers, $x \leq 100$ and $y \leq 100$. Find the minimum number of lines with gradient $+\frac{3}{7}$ we should draw in the way that each *good* point lies on at least one of these lines.

($\rightarrow$ p.9)

Problem 15.

Find the maximum number of bishops that we can place in an 8×8 chessboard in such a way that each bishop threatens at most 3 other bishops.

(Every bishop threatens other pieces in diagonal directions, and if a piece is placed between two bishops, the bishops will not threaten each other anymore.)

($\rightarrow$ p.9)

Answers

Problem	Answer
1	4
2	11
3	2
4	32
5	324
6	8
7	5
8	702
9	900
10	1
11	21
12	9
13	8
14	979
15	40

Second Round

Problems

Problem 1.

In the lake, there are 23 stones arranged along a circle. There are 22 frogs numbered $1, 2, \dots, 22$ (each number appears once). Initially, each frog randomly sits on a stone (several frogs might sit on the same stone). Every minute, all frogs jump at the same time as follows: the frog number i jumps i stones forward in the clockwise direction. (In particular, the frog number 22 jumps 1 stone in the counter-clockwise direction.) Prove that at some point, at least 6 stones will be empty.

($\rightarrow$ p.17)

Problem 2.

We assume a truck as a $1 \times (k+1)$ tile. Our parking is a $(2k+1) \times (2k+1)$ table and there are t trucks parked in it. Some trucks are parked horizontally and some trucks are parked vertically in the parking. The vertical trucks can only move vertically (in their column) and the horizontal trucks can only move horizontally (in their row). Another truck is willing to enter the parking lot (it can only enter from somewhere on the boundary).

1. For $3k+1 < t < 4k$, prove that we can move other trucks forward or backward in such a way that the new truck would be able to enter the lot.
2. Prove that the statement is not necessarily true for $t = 3k+1$.

($\rightarrow$ p.18)

Problem 3.

There is an ant on every vertex of a unit cube. At the time zero, ants start to move across the edges with the velocity of one unit per minute. If an ant reaches a vertex, it alternatively turns right and left (for the first time it will turn in a random direction). If two or more ants meet anywhere on the cube, they die! We know an ant survives after three minutes. Prove that there exists an ant that never dies!

(→ p.20)

Problem 4.

The path number of a graph is the minimum number of paths we need to partition the vertices of a graph. Given a connected graph with the independence number $k > 1$, what is the maximum possible value for the path number in this graph? Find the answer in terms of k .

The independence number of a graph $\mathbf{G}$ is the maximum possible number k , such that there exist k pairwise non-adjacent vertices in $\mathbf{G}$.

(→ p.21)

Problem 5.

By a *tile* we mean a polyomino (i.e. a finite edge-connected set of cells in the infinite grid). There are many ways to place a tile in the infinite table (rotation is allowed but we cannot flip the tile). We call a tile $\mathbf{T}$ *special* if we can place a permutation of the positive integers on all cells of the infinite table in such a way that each number would be maximum between all the numbers that tile covers in at most one placement of the tile.

1. Prove that each square is a special tile.
2. Prove that each non-square rectangle is not a special tile.
3. Prove that tile $\mathbf{T}$ is special if and only if it looks the same after 90° rotation.

(→ p.22)

Problem 6.

Let $\mathcal{P}$ be a convex polygon and $\mathbf{T}$ be a triangle with vertices among the vertices of $\mathcal{P}$. By removing $\mathbf{T}$ from $\mathcal{P}$, we end up with 0, 1, 2, or 3 smaller polygons (possibly with shared vertices) which we call the effect of $\mathbf{T}$. A triangulation of $\mathcal{P}$ is a way of dissecting it into some triangles using some non-intersecting diagonals. We call a triangulation of $\mathcal{P}$ beautiful, if for each of its triangles, the effect of this triangle contains exactly one polygon with an odd number of vertices. Prove that a triangulation of $\mathcal{P}$ is beautiful if and only if we can remove some of its diagonals and end up with all regions as quadrilaterals.

($\rightarrow$ p.25)

Problem 7.

In a group of 2021 people, 1400 of them are *saboteurs*. Sherlock wants to find one saboteur. There are some missions that each needs exactly 3 people to be done. A mission fails if at least one of the three participants in that mission is a saboteur! In each round, Sherlock chooses 3 people, sends them to a mission and sees whether it fails or not. What is the minimum number of rounds he needs to accomplish his goal?

($\rightarrow$ p.27)

Solutions

Problem 1.

In the lake, there are 23 stones arranged along a circle. There are 22 frogs numbered $1, 2, \dots, 22$ (each number appears once). Initially, each frog randomly sits on a stone (several frogs might sit on the same stone). Every minute, all frogs jump at the same time as follows: the frog number i jumps i stones forward in the clockwise direction. (In particular, the frog number 22 jumps 1 stone in the counter-clockwise direction.) Prove that at some point, at least 6 stones will be empty.

Proposed by Josef Tkadlec

Solution.

Consider the first 23 configurations. We say that two frogs collide (during some minute) if they sit on the same stone. Since 23 is a prime, each two frogs collide precisely once during the first 23 configurations. In total, this is $\binom{22}{2} = 231$ collisions.

We will show that during that minute at least 6 stones are empty.

For the sake of contradiction, suppose that at least $23 - 6 + 1 = 18$ stones have a frog on them. For each such stone, name one of its frogs as the captain (of that stone). Hence at most $22 - 18 = 4$ frogs are non-captains. Then during this minute:

- No two captains are colliding.
- Each of the at most 4 non-captains is colliding with 1 captain.
- There are at most $\binom{4}{2} = 6$ collisions among non-captains.

In total, this is at most $4 + 6 = 10$ collision, a contradiction. ■

Problem 2.

We assume a truck as a $1 \times (k+1)$ tile. Our parking is a $(2k+1) \times (2k+1)$ table and there are t trucks parked in it. Some trucks are parked horizontally and some trucks are parked vertically in the parking. The vertical trucks can only move vertically (in their column) and the horizontal trucks can only move horizontally (in their row). Another truck is willing to enter the parking lot (it can only enter from somewhere on the boundary).

1. For $3k+1 < t < 4k$, prove that we can move other trucks forward or backward in such a way that the new truck would be able to enter the lot.
2. Prove that the statement is not necessarily true for $t = 3k+1$.

Proposed by ICO Problem Selection Committee

Solution.

Call the cell in the center of the table O . First we prove that if one of the trucks contains O then there are at most $3k+1$ trucks. This fact concludes the second part easily.

1. For the first part, no truck contains O since $3k+1 < t$. Now we have four types of trucks: D is the set of trucks horizontal and below O , similarly we define the other types U, L and R . We prove that the trucks in D can be moved forward or backward to stand precisely below each other (possibly with some gaps for the rows without truck). Similarly for U, L and R we do the same thing. Then we have four blocks of trucks and we can easily move all of them clockwise or counterclockwise around O to reach a position as in the figure, or its reflection.

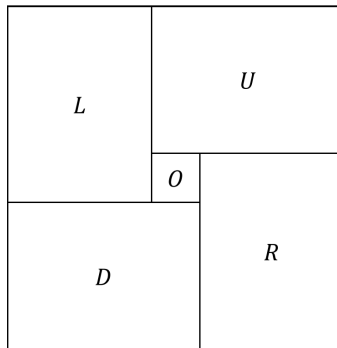
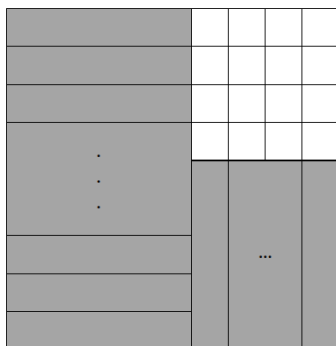


Figure 3: Final position of the trucks

2. For the second part look at the figure below: Assume we have placed some trucks in the gray parts. Number of these trucks are equal to $(2k+1) + k = 3k+1$.



In this figure, There is always a truck which is placed in cell O no matters how we move these trucks. According to the first proof, the number of trucks won't exceed $3k + 1$ so we can not add any truck into this table.

□

Problem 3.

There is an ant on every vertex of a unit cube. At the time zero, ants start to move across the edges with the velocity of one unit per minute. If an ant reaches a vertex, it alternatively turns right and left (for the first time it will turn in a random direction). If two or more ants meet anywhere on the cube, they die! We know an ant survives after three minutes. Prove that there exists an ant that never dies!

Proposed by Morteza Saghafian

Solution.

If an ant does not meet any other ant, then its path would be a hexagon which is symmetric with respect to the center of the cube. So if two ants meet after three minutes, they should have met three minutes before at the opposite points.

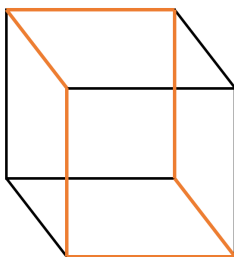


Figure 5: A hexagonal path for ant when it alternatively turns right and left



Problem 4.

The path number of a graph is the minimum number of paths we need to partition the vertices of a graph. Given a connected graph with the independence number $k > 1$, what is the maximum possible value for the path number in this graph? Find the answer in terms of k .

The independence number of a graph $\mathbf{G}$ is the maximum possible number k , such that there exist k pairwise non-adjacent vertices in $\mathbf{G}$.

Proposed by Afrouz Jabal Ameli and Amir Hossein Gorzi

Solution.

We argue that the answer is $k - 1$.

First observe that a star with k leaves, is a connected graph with independence number k and path number $k - 1$.

Now assume that there exists a graph G , such that the *path number* of G is at least k . Consider the minimum number of paths, $t \geq k$, that partition the vertices of G .

If we consider one endpoint per each of these paths, it forms an independent set of vertices, which implies $t \leq k$. Therefore we can assume that $t = k$.

Now, in each of these paths, of length at least 1, such as P , the two endpoints of P must be adjacent in G , otherwise there exists an independent set of size at least $k + 1$ in G . Now, since G is connected there must be an edge e in G between two of these paths, namely P_1 and P_2 . Since for each $i \in \{1, 2\}$, P_i is either a path of length zero or the two endpoints of P_i are adjacent in G , then using the edge e , one can merge these two paths into one path; This is a contradiction since t is the path number of G .

■

Problem 5.

By a *tile* we mean a polyomino (i.e. a finite edge-connected set of cells in the infinite grid). There are many ways to place a tile in the infinite table (rotation is allowed but we cannot flip the tile). We call a tile **T** *special* if we can place a permutation of the positive integers on all cells of the infinite table in such a way that each number would be maximum between all the numbers that tile covers in at most one placement of the tile.

- 1. Prove that each square is a special tile.
- 2. Prove that each non-square rectangle is not a special tile.
- 3. Prove that tile **T** is special if and only if it looks the same after 90° rotation.

Proposed by Morteza Saghafian

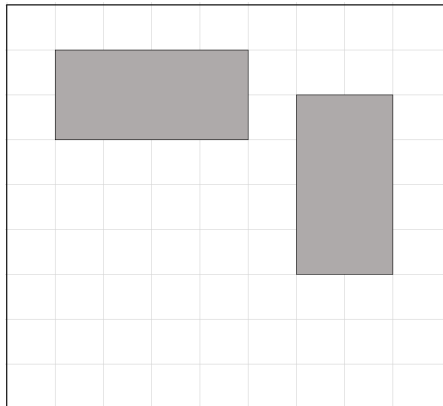
Solution.

- 1. If we place the numbers in the increasing order on a spiral path as in the figure below, then the number on every cell is greater than at most two of its (edge) neighbors. So the number on every cell can be the maximum number on at most one square of the given size.

26	27	28	29	30	31
25	10	11	12	13	32
24	9	2	3	14	33
23	8	1	4	15	34
22	7	6	5	16	$\vdots$
21	20	19	18	17	

Figure 6: All positive integers placed in the increasing order along a spiral path

2. Let the tile T be a $a \times b$ rectangle with $a \neq b$. Consider a large square S with side length N , take all copies of T completely inside S and mark the maximum cell of each of them. The number of copies of T in S is $2 \cdot (N - a + 1) \cdot (N - b + 1)$, since T can also rotate. So the total number of marked cells in S is $2 \cdot (N - a + 1) \cdot (N - b + 1)$, which for large enough N is larger than N^2 . So there exists a cell in S which is marked at least twice, contradiction.



3. If the tile T looks the same after 90° rotation, then the same numbering as part 1 works. Because the numbers on a cell has one of the following cases:
- **Case 1.** It is smaller than all numbers on all the bottom rows and also in its row it is smaller than all numbers left to it (e.g. number 6 in the figure).
 - **Case 2.** It is smaller than all numbers on all the right columns and also in its column it is smaller than all numbers below it (e.g. number 14 in the figure).
 - **Case 3.** It is smaller than all numbers on all the upper rows and also in its row it is smaller than all numbers right to it (e.g. number 12 in the figure).
 - **Case 4.** It is smaller than all numbers on all the left columns and also in its column it is smaller than all numbers above it (e.g. number 9 in the figure).

We assume the first case, the other cases will follow similarly. Suppose that the number on cell c has the property of the first case, and it is the maximum number among all cells of a copy of tile T . Then it has to be the bottom-most cell in this copy of T , and also among the bottom-most cells of this copy, it has to be the leftmost cell. since T looks the same after 90° rotation, we can place a copy of T uniquely in the grid so that cell c is the bottom-most cell and leftmost cell among the bottom-most cells of it.

If the tile T does not look the same after 90° rotation, then the number of copies of T completely inside a large square S of side length N is quadratic in terms of N , with leading coefficient more than 1 (actually 2 or 4). So similar approach to part 2 works.

■

Remark. Note that this solution works even for a finite (not necessarily connected) subset of cells in the grid.

Problem 6.

Let $\mathcal{P}$ be a convex polygon and $\mathbf{T}$ be a triangle with vertices among the vertices of $\mathcal{P}$. By removing $\mathbf{T}$ from $\mathcal{P}$, we end up with 0, 1, 2, or 3 smaller polygons (possibly with shared vertices) which we call the effect of $\mathbf{T}$. A triangulation of $\mathcal{P}$ is a way of dissecting it into some triangles using some non-intersecting diagonals. We call a triangulation of $\mathcal{P}$ beautiful, if for each of its triangles, the effect of this triangle contains exactly one polygon with an odd number of vertices. Prove that a triangulation of $\mathcal{P}$ is beautiful if and only if we can remove some of its diagonals and end up with all regions as quadrilaterals.

Proposed by Abolfazl Asadi

Solution.

We define a quadrilateration of $\mathcal{P}$, a way of dissecting it into some quadrilateral region using some non-intersecting diagonals.

First, We show that if the effects of some triangle T in triangulation has at least two odd polygons, triangulation cannot construct quadrilateration by removing some of the diagonals. Each odd-vertex effect of T must match exactly one of its triangles with T to construct quadrilateration. Therefore, T must be matched with two triangles in the quadrilateration, which is a contradiction.

Now, We prove the other side. We present two solutions for this part:

Solution 1: Proof by induction. We can assume that n (the number of vertices of P) is even (if n is odd, P does not have any quadrilateration). For the basis of induction, $n = 4$ is trivial. Let the statement be true for $k < n$. Now, we show that the statement is also true for $k = n$. Let A be a beautiful triangulation of polygon and T be a triangle of A with two sides coinciding on the sides of the polygon (it's easy to show that every triangulation has such a triangle). Let T' be the adjacent triangle of T . After removing T and T' from A , the remaining part has one or two polygons. These polygons have even sides, because T' has another odd effect (T). In each of these polygons, we have a quadrilateration according to the induction hypothesis (T , T' and other part contain even number of triangles and don't interfere with the induction hypothesis).



Solution 2: Proof by one-to-one correspondence. For every triangle T in triangulation, its correspondence triangle ($f(T)$) determines uniquely (the adjacent triangle in odd effect of T). That's enough to show that $f(f(T)) = T$ for every triangle T in triangulation. This is also trivial, because for an arbitrary triangle T in triangulation, T is in the odd effect of $f(T)$. ■

Problem 7.

In a group of 2021 people, 1400 of them are *saboteurs*. Sherlock wants to find one saboteur. There are some missions that each needs exactly 3 people to be done. A mission fails if at least one of the three participants in that mission is a saboteur! In each round, Sherlock chooses 3 people, sends them to a mission and sees whether it fails or not. What is the minimum number of rounds he needs to accomplish his goal?

Proposed by Afrouz Jabal Ameli

Solution.

The answer is 2346.

Take a person u . Partition the other people into 456 groups of size 3 and 163 groups of size 4. Let these groups be $A_1, \dots, A_{619}$. We do the following algorithm. For each $1 \leq i \leq 619$, For each pair of members of A_i , such as v, w we send v, w and u to a mission until a mission is done successfully or we do all the missions.

If all the 2346 missions fail one can argue that u is a saboteur, otherwise we have at least $456 \times 2 + 163 \times 3 = 1401$ saboteurs which is a contradiction.

So assume that the j -th mission $\{u, v, w\}$ is the first successful mission. Note that j is at most $6 \times 163 + 3 \times 456 = 2346$. If $j = 1$, then clearly we can take the rest of the people one by one and send them to a mission with u and v , until a mission fails and hence we find a saboteur in less than $2021 - 1400$ steps. If $j > 1$, then if one of the previous missions contained v or similarly w , then we already can identify a saboteur in j steps. Otherwise $j < 2346 - 1$. Now by one additional mission, we identify a saboteur.

To show that at least 2346 missions are necessary, assume that it is possible to do this with at most 2345 steps. Now consider the case in which all the missions fail and we decide to present a specific person (u) as a saboteur. Consider all the pairs of people $A_1, \dots, A_r$ that have been sent to a mission with u . Also consider all the sets of size 3 formed by the people who have been sent to a mission together that u did not participate. Let these sets be $B_1, \dots, B_k$. Clearly $k + r = 2345$. Now it suffices to show that there is a set S of 1400 people other than u , such that S intersects $A_1, \dots, A_r, B_1, B_2, \dots, B_k$, which is a contradiction!

To show this let $G = (V, E)$ be the graph, such that V is the set of all the people other than u and E be the set of pairs formed by each A_i and exactly one pair from each B_i . Therefore $|V| = 2020$ and $|E| = 2345$. We show that the minimum vertex cover of this graph is of size at most 1400. To prove this we present an algorithm that obtains a vertex cover of size at most 1400 in G . At each step we take the vertex with the maximum degree, add it to solution, and remove it from the graph, until it has no more edges.

Let $f(m, n)$ be the size of the vertex cover obtained by this algorithm in a graph with n vertices and m edges. Then we have

$$f(m, n) \leq f(m - \Delta, n - 1) + 1 \leq f(m - \lceil \frac{2m}{n} \rceil, n - 1).$$

Thus $f(2345, 2020) \leq 1400$.

■