1st Iranian Combinatorics Olympiad

Contest problems with solutions

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This booklet is prepared by Seyed Reza Hosseini, Matin Yousefi, Benyamin Ghaseminia, Abolfazl Asadi and Alireza Dadgarnia.

With special thanks to Morteza Saghafian.

The 1^{st} Iranian Combinatorics Olympiad was held on April 22^{nd}, 2020 with over 750 participants in 320 groups and each group was consisted of either two or three members. The Problem Selection Committee for 1^{st} Iranian Combinatorics Olympiad was consisted of

> Morteza Saghafian, Alireza Alipour, Yaser Ahmadi Fouladi, Abbas Servati, Abolfazl Asadi, Seyed Reza Hosseini, Seyed Hessam Firouzi, Afrouz Jabalameli

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> Erfan Moeini, Alireza Abdollahpour, Kasra Masoudi, Shayan Peyghambari, Sina Pakseresht, Amirreza Akbari, Arsham Jamshidi, Mohammad Mahdi Hatami, Sadra Dashti, Mohammad Heydari, Mohammad Saeid Haghi, Mohammad Bagher Sharifi, Dorsa Majdi, Mohaddese Ghafari, Sana Amin Naji, Amir Mohammad Imani, Mehrshad Mohammadi, Sogol Samanian, Diba Hashemi

who played a decisive role in holding the contest as online referees. Thanks should also go to Arman Karimi and Mohammad Ali Heydari for designing the online system.

Contents

Short Answer Exam

Problems

1) Consider a 9×9 table with a bear in the middle cell of the top row and a beehive in the middle cell of the bottom row. We want to dig a hole in 6 cells (other than initial cells containing the bear and the beehive) of the table. After digging the holes, the bear starts moving towards the beehive. In each step the bear moves from a cell to a cell with no holes that has an adjacent edge to it. The holes must be chosen such that there is at least a valid path that the bear can take to reach the beehive. The bear always chooses the shortest possible path. In all possible ways of digging the holes, what is the maximum number of steps that the bear takes to reach the beehive?

2) What is the number of sequences of letters a, b and c of length 7 such that no two adjacent letters are the same and no two sub-sequences are equal? A sub-sequence is a consecutive set of letters from the sequence.

3) Consider the ordered pairs $(0,0), (0,1), \ldots, (9,8)$ and $(9,9)$. Assign a card to each of these 100 pairs. We have a subset of these 100 cards and a device that takes two cards like (a, b) and (c, d), then other than returning these cards, it also gives us two more cards with ordered pairs of $(\min\{a, c\}, \min\{b, d\})$ and $(\max\{a, c\}, \max\{b, d\})$. Find the minimum number of cards needed to obtain all 100 cards using the device.

4) Football league organization of Abolfistan has announced that if the Corona pandemic does not end until December, due to shortage of time in the next season every two team will play against each other exactly once. This league has 2048 teams. To choose the host, the organization uses the following algorithm:

> For each match in week i, if the teams who have to play each other have not hosted the same number of games in the past $i-1$ weeks, the host will be the team with lower number of hosting, otherwise the host will be chosen randomly.

What is the maximum possible number of times a team can be a host?

5) Find the total number of possible ways to partition the set $\{1, 2, \dots, 99\}$ into some subsets such that the average of elements in each subset equals to the total number of subsets.

6) For each non-empty subset S of the set $A = \{10, 12, \dots, 26\}$, consider a number c_S defined as

$$c_S = \frac{\text{multiplication of all elements in } S}{2^{|S|}},$$

where $|S|$ is the number of elements of S.

For example, for the subset $S = \{10, 14, 18\}$, we have $c_S = \frac{10\times14\times18}{2^3}$. Find

$$\sum_{S\subseteq A,\ S\neq\emptyset} c_S.$$

7) A frog is at the origin point of the Cartesian plane. Each time, it jumps one or two units to the right. Let $\frac{a}{b}$ be the possibility of the frog getting to the point $(10, 0)$ after a few jumps, where a and b are positive integers and $\gcd(a, b) = 1$. Determine $a + b$.

8) Each diagonal parallel to one of the main diagonals is called a *tape*. For instance, a tape is shown in the following figure:

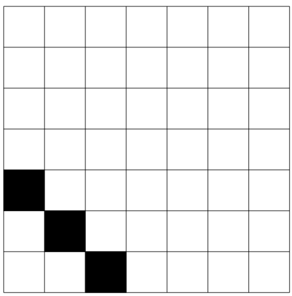


Figure 1: A tape

Note that each corner cell also counts as a tape, thus there are 26 tapes in the given figure. Consider a 1399×1399 table. We want to put some pins in some of the cells of this rectangle such that each tape covers an odd number of pins. Let a and b respectively be the minimum and maximum number of pins to achieve this. Find the value of $a + b$.

9) Determine the maximum number of tiles shown in the figure bellow that can be placed in a 10×10 table such that no two of them share a vertex.

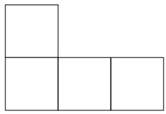


Figure 2: L-shaped tetromino

10) Consider the graph shown in the figure bellow. The *value* of an edge is defined to be the number of edges intersecting it (other than the ending points). The maximum value assigned to edges of a graph is called the *ugliness* of the graph. Asad wants to redraw the given graph in the following figure to minimize its ugliness. What is the minimum ugliness that he can achieve?

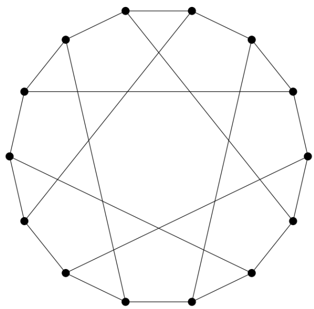


Figure 3: Graph of problem 10

11) Some bids are placed in the cells of a 1399×2020 table. In each turn we can choose a cell with more than one bid, take two of them and place one of them in the cell above and place the other in the cell to the right; If the current cell is the highest cell in its column, the bid is placed in the lowest cell of the column and similarly, a bid moves from the rightmost cell to the leftmost cell of the same row. Determine the minimum number of bids needed such that one can continue this processes forever.

12) We call a set of rectangles as a good set if the following properties are satisfied:

- All sides of the rectangles are either horizontal or vertical.
- The length of the sides of the rectangles can only be from the set $\{1, 2, \dots, 10\}$.
- The length of at least one side of each rectangle is 6.
- The sum of the area of all rectangles is less than 100.

Determine the minimum value of k such that all the rectangles included in an arbitrary good set can be located in a rectangle with width 10 and height k, without overlapping.

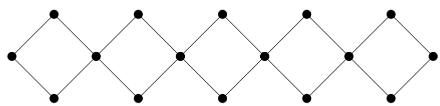
Read the following passage and answer to the next 3 problems:

Suppose that G is a Graph with n vertices labeled by $1, 2, \dots, n$ and A is a subset of vertices of G with even number of elements. We want to choose some edges such that the degree of each vertex in A is odd and the degree of each vertex outside of A is even. Denote the minimum number of such edges by $f(G, A)$.

13) Let $n = 10$ and $A = \{1, 2, 3, 4\}$. For how many initial graphs G we have $f(G, A) = n - 1$?

14) Let G be a cycle with 19 vertices. Find the total sum of $f(G, A)$ for every possible subset A of G.

15) Let G be a graph shown in the following figure.



Find the number of subsets A of G satisfying $f(G, A) = 9$.

Short Answers:

1	16
2	18
3	10
4	1029
5	1
6	726485759
7	1707
8	1958602
9	12
10	1
11	3419
12	18
13	33868800
14	2028478
15	1620

Main Exam

Problems

1- Increasing Difference.

A soccer tournament has 2020 teams. Each pair of teams have played each other exactly once. Suppose that no game have led to a draw. The participating teams are ranked first by their points, 3 points for a win and 0 point for a loss; then by their goal difference which is the number of goals scored minus the number of goals against. Is it possible for the goal difference in such ranking to be strictly increasing from top to bottom?

2- Friendly Game.

Morteza and Amirreza play the following game. First each of them independently rolls a dice 100 times in a row to construct a 100 digit number with digits $\{1, 2, 3, 4, 5, 6\}$; None of these players can see the 100 digit number of the other. After that, each of them simultaneously selects a digit of the other's number. If both selected digits are equal to 6, both players win otherwise they both lose. Is there a playing strategy with more than $\frac{1}{36}$ chance of winning?

3- Intersecting Chords.

On the perimeter of a circle, 1399 points and some chords between them are given.

a) In every step we can take two chords RS, PQ with a common point other than P, Q, R, S and erase exactly one of RS, PQ, then we draw PS, PR, QS and QR(if some of them exist then we just keep the existing chords). Let s be the minimum number of chords after some steps. Find the maximum possible value of s over all possible initial arrangements of chords.

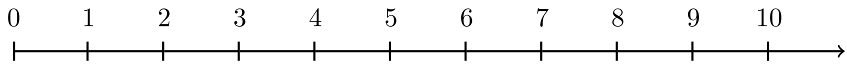
b) In every step we can take two chords RS, PQ with a common point other than P, Q, R, S and erase both RS and PQ and draw PS, PR, QS, QR(if some of them exist then we just keep the existing chords). Let s be the minimum number of chords after some steps. Find the maximum possible value of s over all possible initial arrangements of chords.

4- Common Friend.

At a party there are 99 guests and each person has at least 81 and at most 90 friends. Prove that there exists a group of 10 people with equal number of friends such that they all have a common friend.

5- Way to Success.

The infinite axis shown in the following figure is a view of Abol's way to success!



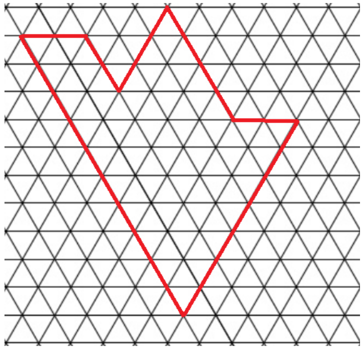
Point 0 refers to his failure. In any other point there is a traffic light. Each light is either blinking in green, yellow or red. Abol is at point 1 and by increasing the number of the point he is in, he gets closer to his goal towards success. In each step the following happens:

> Abol looks at the color of the light in his current point. He will continue moving to the right if the blinking color is green or yellow. If the light blinks red he moves a point to the left; After Abol has moved to either right or left, the light that he previously saw, changes its color from green, yellow and red to yellow, red and green respectively. If Abol sees a red light at point 1 he will resist for the first time and stays in the same point (But the light changes). But the second time he sees a red light in point 1 he gives up and will never move again.

Prove that Abol eventually meets any positive integer point on the axis.

6- Triangular Grid.

Consider a triangular grid of equilateral triangles with unit sides. Assume that $\mathcal{P}$ is a simple polygon with perimeter 1399 and sides from the grid lines. Prove that $\mathcal{P}$ has at least one $120°$ or one $240°$ angle.



a simple polygon in a triangular grid

7- Red Coin.

Seyed has 998 white coins, one red coin and an unusual coin with one red side and one white side. He can not see the color of the coins; Instead he has a scanner which he can place a subset of the coins on one of their sides and the scanner tells him whether all of the coin sides touching its glass are white or not. Prove that Seyed can find the red coin by using the scanner at most 17 times.

Solutions

1- Increasing Difference.

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A soccer tournament has 2020 teams. Each pair of teams have played each other exactly once. Suppose that no game have led to a draw. The participating teams are ranked first by their points, 3 points for a win and 0 point for a loss; then by their goal difference which is the number of goals scored minus the number of goals against. Is it possible for the goal difference in such ranking to be strictly increasing from top to bottom?

Proposed by Abolfazl Asadi

Solution. Assume that such configuration is possible. No two teams can win the Assume that such configuration is possible. No two teams can win the same number of matches, because then the ordering of their goal differences would not satisfy the condition. Each team wins at least 0 and at most 2019 matches. So, the team that finishes k^{th} place wins exactly $2020 - k$ matches. Thus 2020^{th} team always lose, and its goal difference is negative, which implies the goal difference of every team is negative. But it is not possible since the total sum of goal differences equals zero.

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2- Friendly Game.

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Morteza and Amirreza play the following game. First each of them independently rolls a dice 100 times in a row to construct a 100 digit number with digits $\{1, 2, 3, 4, 5, 6\}$; None of these players can see the 100 digit number of the other. After that, each of them simultaneously selects a digit of the other's number. If both selected digits are equal to 6, both players win otherwise they both lose. Is there a playing strategy with more than $\frac{1}{36}$ chance of winning?

Proposed by Morteza Saghafian

Solution. The answer is positive. The strategy is that each player selects the position of his first 6(100 if there is no 6 in his sequence). The possibility that both of them select the same k is $((\frac{5}{6})^k \times \frac{1}{6})^2$. Summing up over all possible ks, we get

$$\sum_{k=1}^{100}((\frac{5}{6})^k \times \frac{1}{6})^2 = \frac{1}{36} + \sum_{k=2}^{100}((\frac{5}{6})^k \times \frac{1}{6})^2 > \frac{1}{36}.$$

3- Intersecting Chords.

On the perimeter of a circle, 1399 points and some chords between them are given.

a) In every step we can take two chords RS, PQ with a common point other than P, Q, R, S and erase exactly one of RS, PQ, then we draw PS, PR, QS and QR(if some of them exist then we just keep the existing chords). Let s be the minimum number of chords after some steps. Find the maximum possible value of s over all possible initial arrangements of chords.

b) In every step we can take two chords RS, PQ with a common point other than P, Q, R, S and erase both RS and PQ and draw PS, PR, QS, QR(if some of them exist then we just keep the existing chords). Let s be the minimum number of chords after some steps. Find the maximum possible value of s over all possible initial arrangements of chords.

Proposed by Afrouz Jabalameli, Abolfazl Asadi

Solution. *a*) Call a chord *side* if it is between two consecutive points and call it a *diameter* otherwise. We prove that the answer is $2n - 3$ whenever there are n points on the circle. Consequently the final answer is 2795.

Consider a convex n-gon along with an arbitrary triangulation of this polygon. Since these chords are pairwise non-intersecting there is no possible moves and the number of initial chords which is $2n - 3$ remains constant.

Now we will prove that for any initial arrangement of chords one can reach a position with no intersecting chords after some number of steps. In that case the number of drawn chords is less than such a number for a triangulation which is $2n-3$. We proceed by induction. The base case is $n=3$ in this case the maximum possible chord is $3=2*3-3$. Now assume the induction hypothesis for all numbers less than n. Note that the claim is true if the initial position has no diameters since there are no possible moves. So we may assume the existence of a diameter like PQ. Now we do the problem's progress on any chord RS intersecting PQ and delete all such RS. Thus after this no chord intersects PQ and we can apply the induction hypothesis on two different sides of PQ. The conclusion fallows.

b)The answer is again $2n-3$ and the construction is similar to A. First we prove by induction that for any $n \geq 4$ if there does not exist intersecting chords then there are two non-adjacent points non of which are an endpoint to a diameter. The base case $n=4$ is obviously true. For the inductive step if there does not exist any diameter the assertion is immediate. So we may assume the existence of a diameter like PQ. Each side of PQ is either a triangle or it has at least four points. In the triangle case the vertex other than P and Q in the triangle has no diameter. When there are four points from the induction hypothesis we can find a vertex with no diameter. In either case there exists a vertex each side of PQ with no diameter drawn from it. These two points apply the desired condition.

Back to the main problem we proceed again by induction and prove that for any initial position one can reach a position with no intersecting chord. For the base case take $n=4$. since there are only 2 diameters assume they intersect. Performing a step on this two intersecting chord will delete all intersections. For the inductive step ignore a vertex A and use the induction hypothesis on the other $n-1$ points. Then there are no intersection and from the first part there are two points with no diameters drawn form them (from the diameters and chords between n-1 points other than A); at least one of these two points is not adjacent with A call that point B. Ignore B and use the inductive step on $n-1$ other points. Observe that every intersection is on AB. If AB does not exist the conclusion follows directly. So assume the existence of AB. Once again form the first part we know that there is a point C other than A which has no diameter (form the second induction points). If C and B are non-adjacent or are adjacent but C is not connected to the other neighbour of B ignoring C from the induction hypothesis on $n-1$ points other than C the result follows. So assume that C is adjacent and connected to the other neighbour of B. After applying a step on intersecting chords AB and CD there is no diameter connected to B. Ignore B and use the induction hypotheses on $n-1$ points other than B deletes all possible intersections.

4- Common Friend.

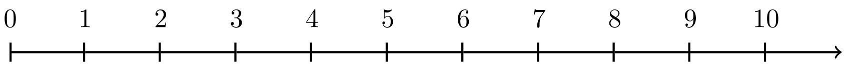
At a party there are 99 guests and each person has at least 81 and at most 90 friends. Prove that there exists a group of 10 people with equal number of friends such that they all have a common friend.

Proposed by Alireza Alipour

Solution. It is impossible for every vertex to have degree 81, because it would imply that the sum of all degrees is odd. Suppose the contrary. If there does not exist such 10 vertices. Let v be a vertex with degree more than 81. Consider $N(v)$, the set of neighbours of v. For each $k \in \{81, 82, \dots, 89\}$, at most 9 vertices of degree k belong to $N(v)$. Since $|N(v)| > 9 \times 9$, there exists a vertex in $N(v)$ of degree 90. Call this vertex w. Consider $N(w)$, the set of neighbours of w. By a similar argument $N(w)$ must contain exactly 9 vertices of degree k for each $k \in \{81, 82, \dots, 90\}$. Hence, there are at least 10 vertices that have degree 90. We claim that they constitute the desired 10 vertices. For each vertex u of the 10 vertices, the number of vertices non-adjacent to u is 9. Hence, there exist at least $99 - 10 \cdot 9 > 1$ vertices adjacent to all of them.

5- Way to Success.

The infinite axis shown in the following figure is a view of Abol's way to success!



Point 0 refers to his failure. In any other point there is a traffic light. Each light is either blinking in green, yellow or red. Abol is at point 1 and by increasing the number of the point he is in, he gets closer to his goal towards success. In each step the following happens:

> Abol looks at the color of the light in his current point. He will continue moving to the right if the blinking color is green or yellow. If the light blinks red he moves a point to the left; After Abol has moved to either right or left, the light that he previously saw, changes its color from green, yellow and red to yellow, red and green respectively. If Abol sees a red light at point 1 he will resist for the first time and stays in the same point (But the light changes). But the second time he sees a red light in point 1 he gives up and will never move again.

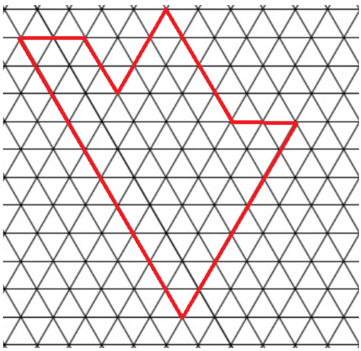
Prove that Abol eventually meets any positive integer point on the axis.

Proposed by Yaser Ahmadi Fouladi

Solution. Assume the contrary. let S be the highest number in which Abol has been at least twice and returned a point back. Note that between any two returns, Abol moves right at least twice. Hence, Abol has passed point S and came back again through the point after S at least twice. It means that Abol has returned back at least twice from the point after S. This is a contradiction to the maximality of S.

6- Triangular Grid.

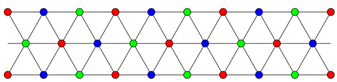
Consider a triangular grid of equilateral triangles with unit sides. Assume that $\mathcal{P}$ is a simple polygon with perimeter 1399 and sides from the grid lines. Prove that $\mathcal{P}$ has at least one $120°$ or one $240°$ angle.



a simple polygon in a triangular grid

Proposed by Seyed Hessam Firouzi

Solution. Suppose the opposite. Color the vertices of the grid in red, green and blue in the following format.



Assume the contrary. Without loss of generality, assume that the first and second color we see are red and blue respectively. Walking around the polygon creates a sequence of colors. In this sequence no two consecutive colors are the same. Also color of two vertices with distance two in the sequence are different since there are no $120°$ or $240°$ angles. So the sequence must be red $\rightarrow$ blue $\rightarrow$ green $\rightarrow$ red $\rightarrow$ blue $\rightarrow$ green $\rightarrow \cdots \rightarrow$ green $\rightarrow$ red. But this means that perimeter of the polygon is divisible by 3, contradicting the assumption.

7- Red Coin.

Seyed has 998 white coins, one red coin and an unusual coin with one red side and one white side. He can not see the color of the coins; Instead he has a scanner which he can place a subset of the coins on one of their sides and the scanner tells him whether all of the coin sides touching its glass are white or not. Prove that Seyed can find the red coin by using the scanner at most 17 times.

Proposed by Seyed Reza Hosseini

Lemma 1: Consider the Fibonacci sequence. Let n be an integer greater than 1, then

$$F_n \leq 2^{n-2}.$$

Proof: We proceed it by induction. For the case $n = 2$, $F_2 = 1 \leq 2^{2-2} = 0$. The inductive step follows from

$$F_{n+1} = F_n + F_{n-1} \leq 2F_n \leq 2.2^{n-2} = 2^{n-1}.$$

Hence the lemma is true for all integers greater than 1.

Lemma 2: We can find the red coin from n coins consisting of a red and $n-1$ white coins in at most $\lceil log_2 n \rceil$ steps.

Proof: We proceed by induction. The base case $n = 1$ is true since there is only one coin and it must be red. Assume that the claim for $i = 1, 2, \ldots n-1$. We prove that it is true for n as well. We take $\lfloor \frac{n}{2} \rfloor$ of the coins and put them in the scanner. From the answer we find out that the red coin is among these coins the remaining ones. So from the induction hypothesis we can find the red coin in at most $\lceil log_2(\lfloor \frac{n}{2} \rfloor) \rceil + 1$ which is less than or equal to $\lceil log_2 n \rceil$ steps.

Back to the main problem. We want to prove that we can find the red coin among any group of at most F_n coins including at most one unusual coin using the scanner at most n times. We proceed by induction. For the Base cases $n = 1$ and $n = 2$ we can find the red coin using the scanner at most 1 and 2 times, respectively. Now assume the claim for $i = 1, 2, \ldots, n-1$. We take F_n coins and divide them into groups A and B with at most F_{n-1} and F_{n-2} coins, respectively. Then we scan group B. We have the following two cases:

1. There is no red side meaning that the red coin is in group A. Thus from the induction hypothesis, we can find it in at most $n-1$ steps.
2. There is a red side. In this case, we reverse the coins in the scanner and scan them again. We have the following two cases:
   (a) There is a red side meaning that the red coin is in this group. From the induction hypothesis we can find the red coin in at most $n-2$ steps.
   (b) There is no red side meaning that group A has a red coin and at most $F_{n-1} - 1$ white coins. From the second lemma can find the red coin in at most $\lceil log_2 F_n - 1 \rceil$ steps. But form the first lemma this number is at most $n-3$ which completes the inductive step.

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